

On Hyperkahler Singularities

A. BELHAJ* and E. H. SAIDI†

High Energy Physics Laboratory, Faculty of sciences , Rabat , Morocco

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Abstract

Using a geometric realization of the $SU(2)_R$ symmetry and a procedure of factorisation of the gauge and $SU(2)_R$ charges, we study the small instanton singularities of the Higgs branch of supersymmetric $U(1)^r$ gauge theories with eight supercharges. We derive new solutions for the moduli space of vacua preserving manifestly the eight supercharges. In particular, we obtain an extension of the ordinary ADE singularities for hyperKahler manifolds and show that the classical moduli space of vacua is in general given by cotangent bundles of compact weighted projective spaces.

*b-adil@emi.ac.ma

†H-saidi@fsr.ac.ma

1 Introduction

Over the few past years, there has been an increasing interest in studying the moduli space of vacua of the Coulomb and Higgs branches of supersymmetric gauge theories with eight supercharges in various dimensions. This interest is mostly due to the fact that supersymmetry severely restricts the quantum corrections to the moduli space metric and allows to make many exact computations [1] . A large class of these gauge theories can be realized by D-brane configurations in type II strings on Calabi Yau manifolds [2, 3, 4, 5] .

Recently, a special interest has been given to the analysis of the hypermultiplet gauge invariant moduli space near the Higgs branch singularity. This analysis has been shown to be relevant for the study of many aspects in supersymmetric gauge theories with eight supercharges amongst which we quote: (1) the understanding of the asymptotic regions of the infrared IR low energy limits of the $N = (4, 4)$ supersymmetric gauge theories in two dimensions especially within the so called throats of the Coulomb and Higgs branches where the theories are typically described by $2d$ $N = 4$ conformal Liouville theories [6, 7]. (2) the study of vector and hypermultiplet moduli spaces in the context of strings compactifications to four dimensions where the low energy supergravity has scalar fields in both vector and matter multiplets [8] . (3) the description of stringy instanton moduli space as hyperkahler deformations of the classical instanton moduli space with non zero NS-NS B field [9, 10, 11]. (4) the rederivation of $2d$ CFT's from compactifications of superstrings on Calabi Yau fourfolds with a $sp(1)$ hyperKahler singularity [12].

The aim of this paper is to study new aspects of hyperkahler singularities of the Higgs branch of supersymmetric gauge theories with eight supercharges by using the linear sigma model method. Actually this study extends the analysis of the ADE singularities of Kahler manifolds to hyperKahler ones. It also generalizes known results, on the Coulomb branch of supersymmetric gauge theories with four supercharges, to the Higgs branch of supersymmetric gauge theories with eight supercharges.

The presentation of this paper is as follows: In section 2, we review hyperKahler singularities and describe briefly the standard way used in handling the D-flatness eq of supersymmetric theories with eight supercharges. In section 3, we use the harmonic superspace method of supersymmetric gauge theories with eight supercharges to study the problem of hyperKahler singularities. This way of doing has many advantages as first it preserves manifestly the eight supercharges second it keeps all the three FI parameters non zero without need to make any appropriate choice. Our method , which is based on realizing the $SU(2)_R$ invariance a by helf

harmonic analysis on S^2 , permits to exhibit explicitly the role of the three Kahler structures of the gauge invariant hyperkahler moduli space. In Section 4, we study the solving of the D-flatness eqs by introducing a method based on the factorization of the charges of the gauge and the $SU(2)_R$ symmetries. We give two classes of solutions of these eqs preserving manifestly the eight supercharges and recovering known results on ADE singularities by particular breaking half of the supersymmetry. In section 5, we give our conclusion.

2 Generalities on Hyperkahler singularities

We begin by noting that Calabi Yau fourfolds can develop singularities of many types; this includes the $\frac{C^4}{Z_4}$ orbifold, the ADE hypersurfaces considered recently in [12] and the so called hyperkahler singularity we are interested in here. We review hereafter the example of the hyperkahler cotangent bundle $T^*(CP^2)$ of complex projective space (CP^2) and its generalisation $T^*(CP^n)$ for $n > 2$. To that purpose, consider $2d$ $N = 4$ supersymmetric $U(1)$ gauge theory with one FI isotriplet coupling $\vec{\xi} = (\xi^1, \xi^2, \xi^3)$ and three hypermultiplets of charges $q_a^i = q^i = 1; i = 1, 2, 3$. The moduli space of this gauge model are obtained by solving

$$\sum_{i=1}^3 [\bar{\varphi}_{i\alpha} \varphi_i^\beta + \varphi_{i\alpha} \bar{\varphi}_i^\beta] = \vec{\xi} \vec{\sigma}_\alpha^\beta, \quad (2.1)$$

which by the way is just a special situation of the following eqs:

$$\sum_{i=1}^n q_a^i [\varphi_i^\alpha \bar{\varphi}_{i\beta} + \varphi_{i\beta} \bar{\varphi}_i^\alpha] = \vec{\xi}_a \vec{\sigma}_\beta^\alpha \quad ; a = 1, \dots, r. \quad (2.2)$$

For later use it is interesting to note that eqs(2.2) have a formal analogy with the sigma model vacuum eqs of $2d$ $N = 2$ supersymmetric $U(1)^r$ gauge theory involved in the analysis of the Coulomb branch of IIA superstrings on Calabi Yau threefolds with ADE singularities.

$$\sum_i q_a^i |X_i|^2 = R_a; \quad a = 1, \dots, r. \quad (2.3)$$

In eqs (2.3), the X_i 's are complex scalar fields, the R_a 's are FI couplings and the q_a^i 's are the $U(1)^r$ gauge charges of the X_i 's which, for reference, read in the case of a $SU(n)$ singularity as :

$$q_a^i = -2\delta_a^i + \delta_a^{i-1} + \delta_a^{i+1}, \quad (2.4)$$

with the remarkable equality

$$\sum_i q_a^i = 0. \quad (2.5)$$

Eqs (2.1) is a system of three eqs which, up to replacing the Pauli matrices by their expressions and using the $SU(2)_R$ transformations $\varphi^\alpha = \varepsilon^{\alpha\beta} \varphi_\beta$ with $\varepsilon_{12} = \varepsilon^{21} = 1$ and $\overline{(\varphi^\alpha)} = \overline{\varphi}_\alpha$, split as follows:

$$\begin{aligned} \sum_{i=1}^3 (|\varphi_i^1|^2 - |\varphi_i^2|^2) &= \xi^3 & (a) \\ \sum_{i=1}^3 \varphi_i^1 \overline{\varphi}_{i2} &= \xi^1 + i\xi^2 & (b) \\ \sum_{i=1}^3 \varphi_i^2 \overline{\varphi}_{i1} &= \xi^1 - i\xi^2 & (c). \end{aligned} \tag{2.6}$$

The moduli space of zero energy states of the classical gauge theory is the space of the solutions of eqs (2.1) and (2.6) divided by the action of the $U(1)$ gauge group. The solutions of eqs (2.1) depend on the values of the FI couplings. For the case where $\xi^1 = \xi^2 = \xi^3 = 0$, the moduli space has an $SU(3) \times SU(2)_R$ symmetry; it is a cone over a seven manifold described by the eqs:

$$\sum_{i=1}^3 (\varphi_{\alpha i} \overline{\varphi}_i^\beta - \varphi_i^\beta \overline{\varphi}_{\alpha i}) = \delta_\alpha^\beta. \tag{2.7}$$

For the case $\vec{\xi} \neq \vec{0}$, the abovementioned $SU(3) \times SU(2)_R$ symmetry is explicitly broken down to $SU(3) \times U(1)_R$. In the remarkable case where $\xi^1 = \xi^2 = 0$ and $\xi^3 > 0$, it is not difficult to see that eqs (2.1) describe the cotangent bundle of CP^2 . Indeed making the change

$$\psi_i = \frac{\varphi_i^1}{[\sum_{i=1}^3 |\overline{\varphi}_{i2}|^2 + \xi^3]^{\frac{1}{2}}}, \tag{2.8}$$

and putting back into eq (2.6.a), one discovers that ψ_i 's satisfy $\sum_i |\psi_i|^2 = 1$. The ψ_i 's parametrize the CP^2 space. On the other hand with $\xi^1 = \xi^2 = 0$ conditions, eqs (2.6-b,c) may be interpreted to mean that $\overline{\varphi}_{2i}$ lies in the cotangent space to CP^2 at the point determined by ψ_i . Note that though we are usually allowed to make the choice $\xi^1 = \xi^2 = 0$ by using an appropriate $SU(2)_R$ transformation, we shall consider in section 4, the generic cases where ξ^1, ξ^2 and ξ^3 are all of them non zero. For the time being let us note that the previous analysis may be extended to the cases of $2d$ $N = 4$ supersymmetric $U(1)$ gauge linear sigma model involving $n + 1$ hypermultiplets with charges $q^i = 1; i = 1, \dots, n + 1$ and transforming in the fundamental representation of $SU(n + 1)$. The vacuum energy equations of this $U(1)$ gauge model read as :

$$\begin{aligned} \sum_{j=1}^{n+1} (|\varphi_j^1|^2 - |\overline{\varphi}_{2j}|^2) &= \xi^3 \\ \sum_{j=1}^{n+1} (\varphi_j^1 \overline{\varphi}_{2j}) &= \xi^1 + i\xi^2. \end{aligned} \tag{2.9}$$

For $\xi^1 = \xi^2 = 0$ and $\xi^3 > 0$, the classical moduli space of the gauge theory is given by the cotangent bundle of complex n projective space: T^*CP^n . For $\xi^1 = \xi^2 = \xi^3 = 0$, one has just the conifold singularity of n dimensional complex manifolds.

3 Hyperkahler singularities in Harmonic superspace

Eqs(2.2) is a system of $3r$ equations which shares some general features with the usual $2d$ $N = 2$ supersymmetric D-flatness conditions eqs(2.3). To solve them, we shall use a different method inspired from harmonic superspace. The latter was shown to be the appropriate space to deal with supersymmetric quantum field theories with eight supercharges. We shall not develop here the basis of this superspace; details may be found in [13, 14]. Our approach will be done in two steps: First we use $SU(2)_R$ harmonic analysis allowing us to realise $SU(2)_R$ representations as special functions on S_R^2 . The index R carried by S_R^2 and $U(1)_R$ refers to the $SU(2)_R$. Second, we introduce a convenient change of variables based on factorizing the $U(1)_G^r$ gauge charges and the $SU(2)_R$ ones. We begin by describing the first step of this program. Introducing the following 2×2 matrix

$$U = \begin{pmatrix} u_1^+ & u_2^+ \\ u_1^- & u_2^- \end{pmatrix} \quad (3.1)$$

and solving the isospin $\frac{1}{2}$ $SU(2)_R$ representation constraints namely the unimodularity $\det U = 1$ and the unitarity $U^\dagger U = U U^\dagger = I$ conditions, one discovers the defining eq of the unit S_R^3 sphere:

$$\begin{aligned} u^{\pm\alpha} &= \epsilon^{\alpha\beta} u_\beta^\pm; & \bar{u}^{+\alpha} &= u_\alpha^-; & \epsilon_{\alpha\beta} &= -\epsilon_{\beta\alpha} \\ u^{+\alpha} u_\alpha^- &= 1, & u^{+\alpha} u_\alpha^+ &= u^{-\alpha} u_\alpha^- = 0. \end{aligned} \quad (3.2)$$

Moreover, using the u_α^\pm variables, the $SU(2)_R$ algebra is realized as differential operators on the space of harmonic functions on S_R^3 :

$$\begin{aligned} D^{++} &= u^{+\alpha} \frac{\partial}{\partial u^{-\alpha}}; & D^{--} &= u^{-\alpha} \frac{\partial}{\partial u^{+\alpha}} \\ 2D^{++} &= [D^0, D^{++}]; & -2D^{--} &= [D^0, D^{--}] \\ D^0 &= [D^{++}, D^{--}] = u^{+\alpha} \frac{\partial}{\partial u^{+\alpha}} - u^{-\alpha} \frac{\partial}{\partial u^{-\alpha}} \end{aligned} \quad (3.3)$$

To study the $SU(2)_R$ representations by using the harmonic variables, it is more convenient to consider harmonic functions $F^q(u_\alpha^\pm)$ with definite $U(1)_R$ charge q ; that is functions $F^q(u_\alpha^\pm)$ satisfying the eigenfunction eq

$$[D^0, F^q] = q F^q. \quad (3.4)$$

These functions F^q have a global harmonic expansion of total charge q and carry $SU(2)_R$ representations. For example, taking $q = 2$ and choosing F^{++} as:

$$F^{++}(u_\alpha^\pm) = u_{(\alpha}^+ u_{\beta)}^+ F^{(\alpha\beta)}, \quad (3.5)$$

one sees that F^{++} is the highest state of the isovector representation of $SU(2)_R$; i.e:

$$[D^0, F^q] = qF^q, \quad [D^{++}, F^q] = 0. \quad (3.6)$$

Note that one can usually realize F^{++} as bilinears of isospinors f^+ and \bar{f}^+ as follows:

$$F^{++} = if^+ \bar{f}^+ = iu_{(\alpha}^+ u_{\beta)}^+ f^{(\alpha} \bar{f}^{\beta)}. \quad (3.7)$$

The complex number i in front of the the factor of the right hand side of the above eqs ensures the reality condition of the isotriplet representation. After this digression on the $SU(2)_R$ harmonic analysis, we turn now to eqs (2.2) which up on multiplying its both sides by $u_{(\alpha}^+ u_{\beta)}^+$, we get:

$$\sum_j q_a^j \varphi_j^+ \bar{\varphi}_j^+ = -i\xi_a^{++}. \quad (3.8)$$

Eqs (3.8) are also the D-flatness eqs one gets if one is using the $2d$ $N = (4, 4)$ harmonic superspace formulation of $2d$ $N = 4$ gauge theories [13, 14].

In the end of this section, we would like to make two comments: First one can use the isospinor bilinear realization of isotriplets eqs (3.7) to represent the Kahler parameters ξ_a^{++} as follows:

$$\begin{aligned} \xi_a^{++} &= i\zeta_a^+ \bar{\zeta}_a^+ = iu_{(\alpha}^+ u_{\beta)}^+ \zeta_a^{(\alpha} \bar{\zeta}_a^{\beta)} \\ \zeta_a^\pm &= u_\alpha^\pm \zeta_a^\alpha; \quad \bar{\zeta}_a^\pm = u_\alpha^\pm \bar{\zeta}_a^\alpha, \end{aligned} \quad (3.9)$$

where ζ_a^α and $\bar{\zeta}_a^\alpha$ may, roughly speaking, be viewed as the square roots of the FI couplings $\xi_a^{(\alpha\beta)}$. Similar relations involving ζ_a^\pm and $\bar{\zeta}_a^\pm$ for ξ_a^0 and ξ_a^{--} may be also written down. Putting back these relations in eqs (3.8), one gets

$$\sum_j q_a^j \varphi_j^+ \bar{\varphi}_j^+ = \zeta_a^+ \bar{\zeta}_a^+ = u_{(\alpha}^+ u_{\beta)}^+ \zeta_a^{(\alpha} \bar{\zeta}_a^{\beta)}. \quad (3.10)$$

Second, simplify further eqs (3.10) by making an extra change of variables which turns out to be convenient when discussing the moduli space of gauge invariant vacua of eqs (2.2). This extra change consists to use the mapping $R^3 = R^+ \times S^2$ to write the FI isovectors ξ_a^{++} as

$$\xi_a^{++} = R_a \eta_a^+ \bar{\eta}_a^+ = r_a^2 \eta_a^+ \bar{\eta}_a^+, \quad (3.11)$$

or equivalently by using the isospinors ζ_a^α and $\bar{\zeta}_a^\alpha$ introduced previously:

$$\begin{aligned} \zeta_a^\pm &= u_\alpha^\pm \zeta_a^\alpha; & \bar{\zeta}_a^\pm &= u_\alpha^\pm \bar{\zeta}_a^\alpha & (a) \\ \zeta_a^\alpha &= r_a \eta_a^\alpha; & \bar{\zeta}_a^\alpha &= r_a \bar{\eta}_a^\alpha & (b) \\ \zeta_a^\alpha \bar{\zeta}_{a\alpha} &= r_a^2 \geq 0 & & & (c). \end{aligned} \tag{3.12}$$

Eqs (3.11) and (3.12) tell us that the R_a 's ($R_a = r_a^2 \geq 0$) are the radial variables and the η_a^α 's and $\bar{\eta}_{a\alpha}$'s, which satisfy

$$\eta_a^\alpha \bar{\eta}_{a\alpha} = 1; \quad \eta_a^\alpha \eta_{a\alpha} = \bar{\eta}_a^\alpha \bar{\eta}_{a\alpha} = 0, \tag{3.13}$$

parametrize the two spheres S_a^2 . The r_a^2 and η_a^α and $\bar{\eta}_{a\alpha}$ are in one to one correspondance with the r FI isovectors. In other words eqs (3.13) describe a collection of r unit two spheres which altogether with the r_a conical variables of $(R^3)^r$ give the $3r$ parameters of the r FI isovector couplings.

4 Solving the D-flatness eqs

In this section, we solve eqs(2.2) by developing a factorization method of the gauge and $SU(2)_R$ charges carried by the hyperKahler moduli. This factorization should satisfy the following natural constraints:

- (a) The splitting of the gauge and $SU(2)_R$ charges preserves the eight supercharges of the gauge theory.
- (b) It recovers the results of [12], extends the ADE models of Kahler backgrounds [3] and, up on breaking half of the eight supercharges, gives the standard ADE results.
- (c) It has a geometrical interpretation.

The factorization of hyperKahler moduli solving the above constraints is given by:

$$\begin{aligned} \varphi_j^+ &= X_j \eta_j^+ + \gamma Y_j \bar{\eta}_j^+ \\ \bar{\varphi}_j^+ &= \bar{X}_j \bar{\eta}_j^+ - \gamma \bar{Y}_j \eta_j^+, \end{aligned} \tag{4.1}$$

where η_j^+ and $\bar{\eta}_j^+$ are as in eqs (3.12); that is:

$$\begin{aligned} \eta_j^+ &= u_\alpha^+ \eta_j^\alpha; & \bar{\eta}_j^+ &= u_\alpha^+ \bar{\eta}_j^\alpha \\ \eta_j^\alpha \bar{\eta}_{\alpha j} &= 1; & \eta_j^\alpha \eta_{\alpha j} &= \bar{\eta}_j^\alpha \bar{\eta}_{\alpha j} = 0. \end{aligned} \tag{4.2}$$

In eqs(4.1), X_j and Y_j , $j = 1, \dots, n$ are complex fields carrying no $SU(2)_R$ charge. The parameter γ takes the values $\gamma = 0$ or $\gamma = 1$ and allow to distinguish two classes of solutions we will give hereafter. Moreover the quantities X_j , Y_j and η_j^+ and $\bar{\eta}_j^+$ of eqs(4.1) behave under

$U(1)^r$ gauge and $SU(2)_R$ transformations as follows:

$$\begin{aligned} U(1)^r : X_j &\longrightarrow X'_j = \lambda^{q_a^j} X_j \\ Y_j &\longrightarrow Y'_j = \lambda^{q_a^j} y_j \\ \eta_j^+ &\longrightarrow \eta'^+_j = \eta_j^+ \\ \bar{\eta}_j^+ &\longrightarrow \bar{\eta}'^+_j = \bar{\eta}_j^+; \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} U(1)_R : X_j &\longrightarrow X'_j = X_j \\ Y_j &\longrightarrow Y'_j = Y_j \\ \eta_j^+ &\longrightarrow \eta'^+_j = e^{i\theta} \eta_j^+ \\ \bar{\eta}_j^+ &\longrightarrow \bar{\eta}'^+_j = e^{i\theta} \bar{\eta}_j^+. \end{aligned} \quad (4.4)$$

Actually eqs (4.3-4) define the factorization of the gauge charges and $SU(2)_R$ ones of the vacuum moduli. In what follows we shall use the splitting eqs(4.1) to solve eqs (2.2) which, by help of the analysis of section 3, may be put in the form:

$$\sum_j q_a^j \varphi_j^+ \bar{\varphi}_j^+ = r_a^2 \eta_a^+ \bar{\eta}_a^+. \quad (4.5)$$

We give herebelow two classes of solutions of these eqs. The first class describes a generalisation of the usual ALE surfaces with ADE singularities. The second class leads to new models which flow in the infrared to $2d$ $N = (4, 4)$ conformal field theories.

4.1 Generalized ADE hypersurfaces

Eqs(2.2) looks formally like eqs (2.3); their solutions are then expected to describe generalisations of the standard eqs of ADE singularities associated with $2d$ $N = 2$ linear sigma models. Imposing the ADE Calabi Yau condition

$$\sum_j q_a^j = 0, \quad a = 1, \dots, n-1, \quad (4.6)$$

one can imitate the analysis of $N = 2$ linear σ models and build the gauge invariant moduli in terms of the φ_j^+ fields. In the $SU(n)$ case for instance where q_a^j is given by eq(2.4), there are three gauge invariant moduli; $U^{+\frac{n(n+1)}{2}}$, $V^{+\frac{n(n+1)}{2}}$ and $Z^{+(n+1)}$ carrying $\frac{n(n+1)}{2}$, $\frac{n(n+1)}{2}$ and $(n+1)$ $U(1)_R$ Cartan charges respectively. These invariant read as:

$$\begin{aligned} U^{+\frac{n(n+1)}{2}} &= \prod_{j=0}^n (\varphi_j^+)^{n-j} \\ V^{+\frac{n(n+1)}{2}} &= \prod_{j=0}^n (\varphi_j^+)^j \\ Z^{+(n+1)} &= \prod_{j=0}^n (\varphi_j^+). \end{aligned} \quad (4.7)$$

They satisfy the following remarkable equation,

$$U^{+\frac{n(n+1)}{2}} V^{+\frac{n(n+1)}{2}} = [Z^{+(n+1)}]^n. \quad (4.8)$$

Eq (4.8) generalizes the usual equation of the ALE surface with $SU(n)$ singularity:

$$uv = z^n. \quad (4.9)$$

To better see the structure of eq (4.8), we use the φ_j^\pm 's moduli factorization described earlier. Taking $\gamma = 0$, the general splitting eqs (4.1) reduces to:

$$\varphi_j^+ = x_j \eta_j^+; \quad \bar{\varphi}_j^+ = \bar{x}_j \bar{\eta}_j^+, \quad (4.10)$$

where X_j and η_j^+ behave under gauge and $SU(2)_R$ transformations as in eqs (4.3-4). Note that like φ_j^\pm , the realization $X_j \eta_j^\pm$ carries for each value of j , four real degrees of freedom; two coming from X_j and the two others from the parameters the sphere S^2 described by η_j^\pm . Under $2d$ $N = 4$ supersymmetric transformations which may be conveniently expressed as $4d$ $N = 2$ supersymmetric transformations of fermionic parameters ϵ^\pm and $\bar{\epsilon}^\pm$, we have:

$$\delta \varphi_j^+ = \epsilon^+ \psi_j + \bar{\epsilon}^+ \bar{\chi}_j; \quad (4.11)$$

where ψ_j and $\bar{\chi}_j$ are the Fermi partners of the φ_j^\pm scalars. $(\varphi_j^\pm, \psi_j, \bar{\chi}_j)$ constitute altogether the $4d$ $N = 2$ free hypermultiplets. Using the splitting principle by factorizing ϵ^+ as $\epsilon \eta^+$ and $\bar{\epsilon}^+ = \bar{\epsilon} \bar{\eta}^+$, and using eqs (4.10-11), we get

$$\eta_j^+ \delta X_j + X_j \delta \eta_j^+ = \bar{\eta}^+ \epsilon \psi_j + \bar{\eta}^+ \bar{\epsilon} \bar{\chi}_j, \quad (4.12)$$

or equivalently

$$\eta_j^\alpha \delta X_j + X_j \delta \eta_j^\alpha = \bar{\eta}^\alpha \epsilon \psi_j + \bar{\eta}^\alpha \bar{\epsilon} \bar{\chi}_j. \quad (4.13)$$

Putting eqs (4.10) back into eqs (4.5), we obtain

$$\sum_j q_a^j |X_j|^2 \eta_j^+ \bar{\eta}_j^+ = r_a^2 \eta_a^+ \bar{\eta}_a^+ \quad (4.14)$$

and

$$\begin{aligned} U^{+\frac{n(n+1)}{2}} &= u M^{+\frac{n(n+1)}{2}} \\ V^{+\frac{n(n+1)}{2}} &= v N^{+\frac{n(n+1)}{2}} \\ Z^{+(n+1)} &= z S^{+(n+1)}; \end{aligned} \quad (4.15)$$

where u, v, z and $M^{+\frac{n(n+1)}{2}}$, $N^{+\frac{n(n+1)}{2}}$ and $S^{+(n+1)}$ are gauge invariants given by:

$$\begin{aligned} u &= \prod_{j=0}^n X_j^j & ; & & N^{+\frac{n(n+1)}{2}} &= \prod_{j=0}^n (\eta_j^+)^{n-j} \\ v &= \prod_{j=0}^n X_j^{n-j} & ; & & M^{+\frac{n(n+1)}{2}} &= \prod_{j=0}^n (\eta_j^+)^j \\ z &= \prod_{j=0}^n X_j & ; & & S^{+(n+1)} &= \prod_{j=0}^n \eta_j^+. \end{aligned} \quad (4.16)$$

Note that u, v and z verify the relation (4.9) and $M^{+\frac{n(n+1)}{2}}$, $N^{+\frac{n(n+1)}{2}}$ and $S^{+(n+1)}$ satisfy eq (4.8). Note moreover that eqs (4.14-15) may be brought to more familiar forms if one requires that all 2-spheres are identified; i.e:

- (i) The $(n+1)$ 2- spheres parametrized by the η_j 's .
 - (ii) The $(n-1)$ η_a 2-spheres used in the parametrization of the FI couplings eqs (3.13).
 - (iii) The η^+ 2-sphere involved in the factorization of the supersymmetric parameter ϵ^+ eq(4.12).
- Put differently, we require the following constraint eq to hold:

$$\eta_j^+ = \eta_a^+ = \eta^+. \quad (4.17)$$

With this identification, eqs (4.14) reduce to the well known D- flatness conditions of the $U(1)^r$ supersymmetric gauge theory with four supercharges; namely:

$$\sum_j q_a^j |X_j|^2 = r_a^2. \quad (4.18)$$

Moreover eq (4.8) reduces to the usual ALE surface with $SU(n)$ singularity (4.9) since the $M^{+\frac{n(n+1)}{2}}$, $N^{+\frac{n(n+1)}{2}}$ and $S^{+(n+1)}$ gauge invariant become trivial; they are given by powers of η^+ as shown here below.

$$\begin{aligned} M^{+\frac{n(n+1)}{2}} &= (\eta^+)^{\frac{n(n+1)}{2}} = N^{+\frac{n(n+1)}{2}} \\ S^{+(n+1)} &= (\eta^+)^{n+1}. \end{aligned} \quad (4.19)$$

In the general case where the gauge charges q_a^j of the X_j 's satisfy the constraints(2.5), eqs(4.18) is the vaccum energy of $2d$ $N = 2$ supersymmetric linear σ models. Thus, the classical moduli space \mathcal{M} of the gauge invariant vacua of eqs(4.17-18) is given by the product of the 2-sphere parametrized by η^+ ; eq(4.17)and the moduli space of the gauge invariant solutions of $2d$ $N = 2$ vaccum energy states. In other words:

$$\mathcal{M} = \frac{C^{n+1}}{C^{*n-1}} \times S^2.$$

Note that the identification constraint eq(4.17) has a nice interpretation; it breaks explicitly half of the eight supersymmetries leaving then four supercharges manifest. These four supercharges are behind the reduction of eqs(4.14) down to eqs(4.18)leading to the standard ADE

models. This feature is immediately derived by combining eqs(4.13) and (4.17) as follows :

$$\eta^\alpha \delta X_j + X_j \delta \eta^\alpha = \bar{\eta}^\alpha \epsilon \psi_j + \bar{\eta}^\alpha \epsilon \bar{\chi}_j. \quad (4.20)$$

Then multiplying both sides of this identity by $\bar{\eta}_\alpha$; one gets, after using eqs(4.2):

$$\delta X_j = \epsilon \psi_j. \quad (4.21)$$

This eq gives the usual supersymmetric transformations of the complex scalars of the $2d$ $N = 2$ chiral multiplets. This completes our check of consistency of the generalised $SU(n)$ hypersurface singularity (4.8). Before going ahead let us summarize in few words what we have been doing. Starting from eqs(2.2), we have shown that it is possible to put them into the equivalent form (4.5). The corresponding moduli space of gauge invariant vacua is given by eq (4.8) which reduces to the standard ALE space with A_{n-1} singularity up on imposing the factorization eqs (4.10) and the conditions (4.17). The latters break four of the original eight supercharges. To restore the eight supersymmetries by still using the constraint (4.17), we should take γ non zero; say $\gamma = 1$. Non zero γ restores four extra supercharges which add to the old four existing ones carried by eq (4.10). Note in passing that a naive analysis of eqs(4.5) suggests that the gauge invariant moduli space of vacua \mathcal{M} of eq (4.10) is given, for the generalized $SU(n)$ singularity eq (4.8), by the usual ALE space with $SU(n)$ singularity times the two-sphere power $2n$:

$$\mathcal{M} = \frac{C^{n+1}}{C_*^{n-1}} \times (S^2)^{2n},$$

where $(n+1)$ two spheres come from the φ_j^+ 's as shown in eqs (4.10) and $(n-1)$ two spheres come from the FI couplings.

4.2 Second Solution

Putting $\gamma = 1$ in eqs(4.1) and (4.5), we get the system of three eqs given by:

$$\begin{aligned} \sum_j q_a^j (|X_j|^2 - |Y_j|^2) \eta_j^+ \bar{\eta}_j^+ &= r_a^2 \eta_a^+ \bar{\eta}_a^+ & (a) \\ \sum_j q_a^j (X_j \bar{Y}_j) \eta_j^+ \eta_j^+ &= 0 & (b) \\ \sum_j q_a^j (\bar{X}_j Y_j) \bar{\eta}_j^+ \bar{\eta}_j^+ &= 0. & (c) \end{aligned} \quad (4.22)$$

At this level no constraint on the FI couplings has been imposed yet. If moreover we require that all the two sphere η_j^+ and η_a^+ are identified as in eq(4.17); the above system reduces to:

$$\begin{aligned} \sum_j q_a^j (|X_j|^2 - |Y_j|^2) &= r_a^2 & (a) \\ \sum_j q_a^j (X_j \overline{Y}_j) &= 0 & (b) \\ \sum_j q_a^j (\overline{X}_j Y_j) &= 0. & (c) \end{aligned} \tag{4.23}$$

Eqs(4.22) have remarkable features which have nice interpretations. Though the q_a^j gauge charges of the hypermultiplet moduli are not required to add to zero as in eq (2.5), eq (4.22.a) behave exactly as the D-flatness condition of $2d$ $N = 4$ supersymmetric $U(1)^r$ gauge theory. The point is that eqs(4.22.a) involve twice the number of fields of eqs(2.3), but with opposite charges q_a^j . Put differently; eq(4.22) involve two sets of fields X_j and Y_j of charge q_a^j and $(-q_a^j)$ respectively. The sum of gauge charges of the X_j 's and Y_j 's add automatically to zero eventhough eq(2.5) is not fulfilled.

$$\sum_j q_a^j + \sum_j (-q_a^j) = 0. \tag{4.24}$$

Eqs(4.24) go beyond the constraint eqs(2.5) and so models with $\gamma = 1$ flow in the IR to $2d$ $N = (4, 4)$ superconformal models extending the usual $2d$ $N = (2, 2)$ ADE ones. Moreover eqs(4.24) may be fulfilled in different ways; either by taking all charges q_a^j of the $U(1)^r$ gauge theory to be positive; say $q_a^j = 1; a = 1, \dots, r; j = 1, \dots, n$, or part of the q_a^j 's are positive and the remaining ones are negative. In the case of a supersymmetric $U(1)$ gauge theory with $(n + 1)$ hypermutiplets with gauge charges equal to one, eqs(4.23.a) describe a CP^n manifold while eqs(4.23-b,c) which read as

$$\sum_j X_j \overline{Y}_j = 0, \tag{4.25}$$

together with their complex conjugate, show that the \overline{Y}_j 's are in the cotangent space of CP^n at the point $x_j = X_j / [\sum_i |Y_i|^2 + r_a^2]^{-\frac{1}{2}}$. Observe that in case where some of the positive charges q_a^j of $U(1)^r$ gauge theory are not equal to one, the corresponding moduli space is just the cotangent bundle of some weighted complex projective space, $T^*(WP^n)$. Obesrve moreover that in the infrared limit this gauge theory flows to a $2d$ $N = (4, 4)$ conformal field theory with central charge $C = 6n$.

5 Conclusion

In this paper we have studied the solutions of the D-flatness eqs of supersymmetric $U(1)^r$ gauge theories with eight supercharges by using the harmonic superspace linear sigma model

approach . We have studied the blown up of hyperkahler singularities and shown that they are given by cotangent bunbles of compact weighted projective spaces. The latters depend on the number of hypermultiplets and gauge supermultiplets involved in the gauge model one is considering. This examination extends the standard linear sigma model analysis performed for the Kahler Coulomb branch of supersymmetric gauge theories with four supercharges. Our method, which go beyond standard analysis where only half of the eight supersymmetries are apparent, preserves manifestly all the eight supersymmetries and is done in two steps. First we have used a geometric realization of the $SU(2)_R$ symmetry to transform the D-flatness eq in a more convenient form quite easy to handle. Second we have introduced a method of factorization of gauge and $SU(2)_R$ charges of the hypermultiplet moduli. This factorisation involves an index γ taking the values $\gamma = 0$ or $\gamma = 1$ which allow to distinguish two classes of solutions of eqs(2.2)preserving the eight supercharges. For $\gamma = 0$, we have obtained a generalisation of the ADE complex surfaces which read in the non affine case as:

$$\begin{aligned}
A_{n-1} : & \quad U^{+\frac{n(n+1)}{2}} V^{+\frac{n(n+1)}{2}} = [Z^{+(n+1)}]^n \\
D_n : & \quad (x^{++})^n + x^{++}(y^{+(n-1)})^2 + (z^{+n})^2 = 0 \\
E_6 : & \quad (x^{+6})^2 + (y^{+4})^3 + (z^{+3})^4 = 0 \\
E_7 : & \quad (x^{+9})^2 + (y^{+6})^3 + y^{+6}(z^{+4})^3 = 0 \\
E_8 : & \quad (x^{+15})^2 + (y^{+10})^3 + (z^{+6})^5 = 0.
\end{aligned} \tag{5.1}$$

The above eqs reproduce the standard ADE singularities by partial breaking of $2d \ N = 4$ supersymmetry down to $2d \ N = 2$. For $\gamma = 1$, we have found new models which flow in the infrared to $2d \ N = (4, 4)$ scale invariant models with central charge $c = 6k$.

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